A NEW CLASS OF DEFORMED SPECIAL FUNCTIONS FROM QUANTUM HOMOGENEOUS SPACES.

F.Bonechi [1], *R.Giachetti* [1], *M.A.del Olmo* [2], *E.Sorace* [1] *and M.Tarlini* [1].

[1]Dipartimento di Fisica, Università di Firenze e INFN Sezione di Firenze, Italy.
[2]Departamento de Física Teórica, Universitad de Valladolid E-47011, Spain.

Abstract. We study the most elementary aspects of harmonic analysis on a homogeneous space of a deformation of the two-dimensional Euclidean group, admitting generalizations to dimensions three and four, whose quantum parameter has the physical dimensions of a length. The homogeneous space is recognized as a new quantum plane and the action of the Euclidean quantum group is used to determine an eigenvalue problem for the Casimir operator, that constitutes the analogue of the Schrödinger equation in the presence of such deformation. The solutions are given in the plane wave and in the angular momentum bases and are expressed in terms of hypergeometric series with non commuting parameters. PACS: 02.20+b, 03.65.Fd, 11.30-j

1. Introduction.

Homogeneous spaces provide a unified framework for a wide class of mathematical problems and often give a sound geometrical interpretation to many results of classical and quantum physics. The definition of special functions, the integral transformations and the harmonic analysis are significant instances in the context of pure mathematics; the classification of elementary Hamiltonian systems by means of coadjoint orbits and their quantization according to the Kirillov theory [1] is one of the most important applications to physical problems. The homogeneous spaces of kinematical groups, such as the Euclidean or the Poincaré group, moreover, originate in a natural way the fundamental wave equations of mathematical physics: indeed these equations are determined by an invariant element of the corresponding Lie algebra operating with a canonical action on the functions on the homogeneous space. It appears therefore that homogeneous spaces are a major constituent of the theory of Lie groups and their applications.

As soon as the theory of quantum groups was founded, it seemed natural to introduce the definition of quantum homogeneous spaces. Since, in this case, one could not apply to the geometrical notion of underlying manifold, the approach was necessarily algebraiuantum spheres were initially defined [2]; later on a systematic work of generalization of the procedure and a detailed study of possible applications to special functions was undertaken [3-9]. One could observe that the largest efforts were devoted to homogeneous spaces of compact quantum groups: no surprise in that, since that part of the theory was best known.

As mentioned above, in applications of physical nature a central position is occuped by kinematical symmetries described, for instance, by the Heisenberg and the Euclidean or pseudo-Euclidean groups. A large number of papers has dealt with the problem of defining the q-deformations of these groups, starting with the Heisenberg [10-12] and with the Euclidean group in two dimensions, $E(2)$ [10]. The situation for the latter has been clarified in [13], where it has been shown the possibility of two different deformations of $E(2)$. The first one, that we denote by $E_q(2)$, has been the most studied [14-23]: its enveloping algebra has the same relations as in the classical case, while the coproducts of the 'translation' generators are twisted primitive with respect to the exponential of the 'rotation' generator. The second deformation, that we denote by $E_\ell(2)$, obtained by a simple contraction procedure [24-26], was later on generalized to dimensions three [27-28] and four [29-30], producing, *e.g* the so called κ-Poincaré. Its main feature is that the deformation parameter undergoes a contraction and acquires physical dimensions: by means of an appropriate rescaling of the generators the parameter can be reabsorbed, as was to be expected on a physical ground. We thus get a deformation of $E(2)$ without parameter: the singular nature of this deformation has been fully clarified in [13].

In a recent paper [31] we have studied the homogeneous spaces of $E_q(2)$. We have found two structures that were already introduced in literature from different points of view and independently of the action of $E_q(2)$, namely 'quantum planes' [32,33] and 'quantum hyperboloids' [34]. Moreover, by quantizing some Poisson homogeneous spaces, also a 'quantum cylinder' has been obtained [35]. The existence of the Haar functional has been proved for $E_q(2)$, [36,37]: by projecting it on homogeneous spaces, the techniques of the usual harmonic analysis can be extended to the realm of quantum homogeneous spaces and the connection with q-special functions can be made explicit. From a physical point of view, this makes conceivable the study of the solutions of explicit models along the usual lines of wave mechanics.

In this paper we define a new quantum plane as a quantum homogeneous space of $E_\ell(2)$. We then specify the canonical action of $E_\ell(2)$ on this space: according to the lines developed in [31], the action will be used to define an eigenvalue equation for the Casimir of $E_\ell(2)$, that constitutes a new deformed version of the free Schrödinger equation. Due to the absence of the deformation parameter, the results we are going to present are of a deeply different nature with respect to those obtained by the similar analysis developed in [31]: there, the Hahn-Exton functions and the q-exponentials are recovered as q-deformation of the Bessel and exponential functions. In the present case, the diagonalization of the Casimir on the linear and angular momentum bases yields new special functions that can be expressed in terms of hypergeometric series with non commuting parameters. The reasons because we find it interesting to present our results are twofold. In the first place this type of harmonic analysis can turn out to be relevant, or even fundamental, for the solution of any possible model presenting such a quantum group symmetry. For instance, a different real form of this group – reproducing a 1+1 deformation of the Poincaré group – has been fruitfully applied to phonon physics and has proved to be the dynamical symmetry group for such a physical system on the lattice [38]. Physical systems with similar properties have been studied in [39,40]. Secondly, since the deformation of $E_\ell(2)$ has been extended to higher dimensions, our results certainly provide a very useful support for understanding the nature of the special functions on those extensions.

2. Quantum homogeneous spaces of $E_\ell(2)$.

In this section we define the homogeneous spaces of $E_\ell(2)$ that will be used in the following analysis. In order to make the presentation sufficiently self-consistent, we find it useful to recall in a sketchy way the algebraic properties of $E_\ell(2)$, as obtained in [10] and its duality with the quantized functions, found in [13].

(2.1) Definition. The Hopf algebra generated by $e^{-i\,\theta}$, a_1, a_2, with relations

$$[e^{-i\,\theta}, a_1] = \frac{z}{2}\,(1-e^{-i\,\theta})^2\,, \quad [e^{-i\,\theta}, a_2] = i\,\frac{z}{2}\,(e^{-2i\,\theta}-1)\,,$$
$$[a_1, a_2] = i\,z\,a_1\ ,$$

coalgebra operations

$$\begin{aligned}\Delta(e^{-i\,\theta}) &= e^{-i\,\theta}\otimes e^{-i\,\theta}\,,\\ \Delta(a_1) &= \cos(\theta)\otimes a_1 - \sin(\theta)\otimes a_2 + a_1\otimes 1\,,\\ \Delta(a_2) &= \sin(\theta)\otimes a_1 + \cos(\theta)\otimes a_2 + a_2\otimes 1\,,\end{aligned}$$

and antipode

$$S(a_1) = -\cos(\theta)\, a_1 - \sin(\theta)\, a_2 \,, \qquad S(e^{-i\,\theta}) = e^{i\,\theta} \,,$$
$$S(a_2) = \sin(\theta)\, a_1 - \cos(\theta)\, a_2 \,,$$

will be called the *algebra of the quantized functions on* $E(2)$ and denoted by $\mathcal{F}_\ell(E(2))$.

Assuming z real, a compatible involution is given by

$$a_1^* = a_1 \,, \quad a_2^* = a_2 \,, \quad \theta^* = \theta \,.$$

The quantized enveloping algebra $\mathcal{U}_\ell(E(2))$ is generated by the unity and the three elements $P_1\,, P_2\,, J$ satisfying

$$[J, P_1] = (i/z)\sinh(zP_2)\,, \quad [J, P_2] = -i\; P_1\,, \quad [P_1, P_2] = 0\,,$$

and such that

$$\Delta P_1 = e^{-z\,P_2/2} \otimes P_1 + P_1 \otimes e^{z\,P_2/2}, \quad \Delta P_2 = P_2 \otimes 1 + 1 \otimes P_2\,,$$

$$\Delta J = e^{-z\,P_2/2} \otimes J + J \otimes e^{z\,P_2/2},$$

$$S(P_2) = -P_2\,, \qquad S(P_1) = -P_1\,, \qquad S(J) = -J - i\,z\,P_1/2\,,$$

with vanishing counity and involution

$$J^* = J\,, \qquad P_1^* = P_1\,, \qquad P_2^* = P_2.$$

We finally write the duality pairing between $\mathcal{U}_\ell(E(2))$ and $\mathcal{F}_\ell(E(2))$,

$$\langle \nu_1, \theta^r a_1^s a_2^t \rangle = \delta_{r,0}\delta_{s,1}\delta_{t,0}\,, \quad \langle \nu_2, \theta^r a_1^s a_2^t \rangle = \delta_{r,0}\delta_{s,0}\delta_{t,1}\,,$$
$$\langle \tau, \theta^r a_1^s a_2^t \rangle = \delta_{r,1}\delta_{s,0}\delta_{t,0}\,,$$

where

$$\tau = -i\, e^{-z\,P_2/2}\,(J - i(z/4)\,P_1),$$

and

$$\nu_1 = -i\, e^{-z\,P_2/2}\,P_1\,, \qquad\qquad \nu_2 = -i\,P_2\,.$$

Observing that $\Delta\tau = e^{-iz\,\nu_2} \otimes \tau + \tau \otimes 1$ and using the condition $\langle u^*, a\rangle = \overline{\langle u, (S(a))^*\rangle}$, with $u \in \mathcal{U}_\ell(E(2))$ and $a \in \mathcal{F}_\ell(E(2))$, we have $\nu_1^* = -\nu_1$, $\nu_2^* = -\nu_2$,

$\tau^* = -\tau - i\,z\,\nu_1$. Moreover, if we consider the rescaled variables $z\,P_1\,,\; z\,P_2 \in \mathcal{U}_\ell(E(2))$ and $a_1/z,\; a_2/z \in \mathcal{F}_\ell(E(2))$, we see that the deformation parameter is reabsorbed: this means that all the structures corresponding to different values of z are isomorphic among themselves. It is nevertheless useful to maintain explicitly the deformation parameter in order to perform more easily the classical limit ($z \to 0$).

We recall the general definition of the two different left actions of an element Y of the quantized enveloping algebra on a quantized function f, namely

$$\ell(Y)f = (id \otimes Y) \circ \Delta f = \sum_{(f)} f_{(1)} \,\langle Y, f_{(2)} \rangle\,,$$
$$\lambda(Y)f = (S(Y) \otimes id) \circ \Delta f = \sum_{(f)} \langle S(Y), f_{(1)} \rangle\, f_{(2)}\,.$$

For later use (and with obvious notation) we also recall that these actions have the following properties:

$$\ell(YZ)f = \ell(Y)\ell(Z)\,f\,, \qquad \lambda(YZ)f = \lambda(Y)\lambda(Z)\,f,$$

and

$$\ell(Y)fg = \sum_{(Y)} \ell(Y_{(1)})f\;\ell(Y_{(2)})g\,, \qquad \lambda(Y)fg = \sum_{(Y)} \lambda(Y_{(2)})f\;\lambda(Y_{(1)})g\,.$$

Following the theory developed in [41], based on the "infinitesimal invariance" of the functions on the quantum homogeneous spaces, we can use the approach of [31] to look for quantum homogeneous spaces of $E_\ell(2)$.

(2.2) Lemma. *Define* $X = J - i(z/4)\,P_1$. *The linear span of* X *constitutes a* $(* \circ S)$*-invariant two-sided coideal of* $\mathcal{U}_\ell(E(2))$*, twisted primitive with respect to* $e^{-z\,P_2/2}$.

Proof. By a straightforward calculation we have

$$* \circ S\,(X) = -X\,, \quad \text{and} \quad \Delta X = e^{-z\,P_2/2} \otimes X + X \otimes e^{z\,P_2/2}\,. \;\blacksquare$$

(2.3) Proposition. *Let* $x = a_1 - i\,a_2\,,\ \bar{x} = a_1 + i\,a_2\,.$ *Then* $x^* = \bar{x}$ *and*

$$[x, \bar{x}] = -z\,(x + \bar{x})\,,$$

Moreover x and $\bar{x}$ generate the invariant subalgebra and left coideal

$$B_X = \{f \in \mathcal{F}_\ell(E(2)) | \ell(X)\, f = 0\}\,.$$

They thus define a quantum homogeneous space whose coaction reads:

$$\delta x = e^{-i\,\theta} \otimes x + x \otimes 1\,, \qquad \delta \bar{x} = e^{i\,\theta} \otimes \bar{x} + \bar{x} \otimes 1\,.$$

Proof. First we observe that $X = i\, e^{z\, P_2/2}\, \tau$, so that the kernel of $\ell(X)$ is the same as that of $\ell(\tau)$. Let us write $f = \sum_{l,m,n} f_{lmn}\, e^{-il\,\theta} a_1^m a_2^n$. Then

$$\begin{aligned}\ell(\tau)\, f &= \sum_{l,m,n} f_{lmn} \left(\ell(e^{-i\,z\,\nu_2})\, e^{-il\,\theta} \ell(\tau) a_1^m a_2^n + (\ell(\tau)\, e^{-il\,\theta}) a_1^m a_2^n\right) \\ &= -\, i \sum_{l,m,n} l\, f_{lmn}\, e^{-il\,\theta} a_1^m a_2^n\end{aligned}$$

that vanishes for $l = 0$. Then the space B_X is generated by a_1 and a_2 or by x and $\bar{x}$. It is immediate to calculate the relationships between x and $\bar{x}$ as well as the coactions. ■

(2.4) Proposition. *The action λ on $\mathcal{F}_\ell(E(2))$ has the following form:*

$$\begin{aligned}&\lambda(P_1)\, e^{-il\,\theta} a_1^m a_2^n = -i\, m\, e^{-il\,\theta} a_1^{m-1} (a_2 + i\, z/2)^n\,, \\ &\lambda(P_2)\, e^{-il\,\theta} a_1^m a_2^n = -i\, n\, e^{-il\,\theta} a_1^m a_2^{n-1}\,, \\ &\lambda(J)\, e^{-il\,\theta} a_1^m a_2^n = i\, e^{-il\,\theta} \Big(\big(il\, a_1^m + m\, a_1^{m-1} (a_2 - \frac{i}{2}(m - 1/2) z)\big)(a_2 + i\, z/2)^n \\ &\qquad\qquad + \frac{i}{2z} a_1^{m+1} \big((a_2 + i\, z/2)^n - (a_2 - i\, 3z/2)^n\big)\Big)\,.\end{aligned}$$

In particular

$$\begin{aligned}\lambda(X)\, e^{-il\,\theta} a_1^m a_2^n = i\, e^{-il\,\theta} \Big(&\big(il\, a_1^m + m\, a_1^{m-1} (a_2 - imz/2)\big)(a_2 + i\, z/2)^n \\ &+ \frac{i}{2z} a_1^{m+1} \big((a_2 + i\, z/2)^n - (a_2 - i\, 3z/2)^n\big)\Big)\,.\end{aligned}$$

Proof. A straightforward calculation. ■

3. Free ℓ-Schrödinger equation.

The natural ℓ-analog of the free Schrödinger equation is obtained by considering the canonical action of the Casimir of $\mathcal{U}_\ell(E(2))$ on the homogeneous spaces so far determined.

The Casimir of $\mathcal{U}_\ell(E(2))$ reads

$$\mathcal{C} = 4H^+H^- = (4/z^2)\ \sinh^2((z/2)\,P_2) + P_1^2\,,$$

where the elements

$$H^+ = \frac{1}{2z}(e^{z\,P_2}-1) - \frac{1}{2}i\,e^{z\,P_2/2}\,P_1\,,\quad H^- = \frac{1}{2z}(1-e^{-z\,P_2}) + \frac{1}{2}i\,e^{-z\,P_2/2}\,P_1$$

are the deformations of the holomorphic and antiholomorphic operators $P_2/2 \mp iP_1/2$. The coproducts of $H^\pm$ are

$$\Delta H^+ = 1\otimes H^+ + H^+\otimes e^{z\,P_2}\,,\qquad \Delta H^- = e^{-z\,P_2}\otimes H^- + H^-\otimes 1\,.$$

Thus the z-deformed free Schrödinger equation reads:

$$4\,\lambda(H^+H^-)\,\psi = E\,\psi\,. \tag{3.1}$$

In the remaining part of this section we shall diagonalize the operator on the right hand side of (3.1) in the 'plane wave' and 'angular momentum' bases, in analogy to the usual procedure carried on in quantum mechanics. For later convenience we recall the definition of the *Pochammer symbol*

$$(a)_n = \prod_{k=0}^{n-1}(a+k)$$

and of the classical generalized hypergeometric series

$${}_rF_s\left[\begin{matrix} a_1 & \cdots & a_r \\ b_1 & \cdots & b_s \end{matrix}\ ;\ \zeta\right] = \sum_{\ell=0}^{\infty}\frac{(a_1)_\ell\cdots(a_r)_\ell}{\ell!\,(b_1)_\ell\cdots(b_s)_\ell}\,\zeta^\ell\ .$$

We also find it useful to introduce the variables

$$\chi = x/z\,,\qquad\qquad \bar\chi = -\bar x/z\,.$$

(*i*) *The plane wave states.*

In analogy with the standard procedure of diagonalization of the Schrödinger operator on the plane wave basis, it is natural to consider H^+ and H^- as the appropriate candidates to be diagonalized. We first determine the action of these operators.

(3.2) LEMMA. *The following relations hold:*

$$\lambda(H^+)\,(\bar{\chi})_n = 0\,, \qquad \lambda(H^+)\,(\chi)_n = -\frac{n}{z}\,(\chi)_{n-1}\,,$$
$$\lambda(H^+)\,(1-\chi)_n = -\frac{n}{z}\,(1-\chi)_n/(1-\chi)\,,$$

and

$$\lambda(H^-)\,(\chi)_n = 0\,, \qquad \lambda(H^-)\,(\bar{\chi})_n = \frac{n}{z}\,(\bar{\chi})_n/(\bar{\chi})\,,$$
$$\lambda(H^-)\,(1-\bar{\chi})_n = \frac{n}{z}\,(1-\bar{\chi})_{n-1}\,.$$

Proof. By direct computation. ∎

These relations suggest to look for plane wave states of the form

$$\psi_{h^+h^-} = \sum_{m,n} h_{mn}\,(\chi)_n\,(1-\bar{\chi})_m\,,$$

where h_{mn} are to be found from the equation

$$\lambda(H^+)\,\psi_{_{h^+h^-}} = h^+\,\psi_{_{h^+h^-}}\,.$$

(3.3) PROPOSITION. *The coefficients h_{mn} are determined up to a multiplicative constant and have the following expression*

$$h_{mn} = \frac{1}{m!n!}(-zh^+)^m\,(zh^-)^n\,.$$

Proof. Indeed, using lemma (3.2),

$$\lambda(H^+)\,(\chi)_m(1-\bar{\chi})_n = -\frac{m}{z}(\chi)_{m-1}(1-\bar{\chi})_n\,,$$
$$\lambda(H^-)\,(\chi)_m(1-\bar{\chi})_n = \frac{n}{z}(\chi)_m(1-\bar{\chi})_{n-1}\,,$$

so that

$$\lambda(H^+)\,\psi_{_{h^+h^-}} = -\frac{1}{z}\sum_{m,n}(m+1)\,h_{m+1,n}(\chi)_m(1-\bar\chi)_n\,,$$

$$\lambda(H^-)\,\psi_{_{h^+h^-}} = \frac{1}{z}\sum_{m,n}(n+1)\,h_{m,n+1}(\chi)_m(1-\bar\chi)_n\,,$$

We thus find the following recurrence relations

$$-\frac{1}{z}(m+1)h_{m+1,n} = h^+\,h_{mn}\,, \qquad \frac{1}{z}(n+1)h_{m,n+1} = h^-\,h_{mn}\,,$$

whose solution is straightforward and yields the result. ■

The Casimir is obviously diagonal on these states:

$$\lambda(\mathcal{C})\,\psi_{_{h^+h^-}} = 4\,\lambda(H^+H^-)\,\psi_{_{h^+h^-}} = 4\,h^+h^-\,\psi_{_{h^+h^-}}\,.$$

Finally, the eigenfunctions $\psi_{_{h^+h^-}}$ can be expressed in the form of a hypergeometric series. Indeed

$$\begin{aligned}\psi_{_{h^+h^-}} &= \sum_{m=0}^{\infty}\frac{(-zh^+)^m}{m!}(\chi)_m\sum_{n=0}^{\infty}\frac{(zh^-)^n}{n!}(1-\bar\chi)_n\\ &= (1+zh^+)^{-\chi}(1-zh^-)^{\bar\chi}-1\end{aligned}$$

and, according to the general definition, this can be written as

$$\psi_{_{h^+h^-}} = {}_1F_0\left[\begin{matrix}\chi\\ -\end{matrix}\;;\;-z\,h^+\right]\;{}_1F_0\left[\begin{matrix}1-\bar\chi\\ -\end{matrix}\;;\;z\,h^-\right]\,.$$

In the classical limit $z\to 0$ we recover the usual plane waves $e^{i\,(h^+x-h^-\bar x)}$, as expected.

(ii) The angular momentum states.

We now consider the diagonalization of (3.1) on a basis which realizes the deformed counterpart of the angular momentum states. The duality structure and the requirement of a correct behaviour under the involution indicate that such a result can be obtained by diagonalizing the operator

$$\mathcal{J} = e^{-z\,P_2/2}(J - i(z/4)\,P_1).$$

Observing that $\mathcal{J}^* = \mathcal{J}$, let us therefore discuss the equation (3.1) together with

$$\lambda(\mathcal{J})\,\psi = r\,\psi\,.$$

We now prove the following two lemmas.

(3.4) LEMMA. *The polynomials* $(\chi)_n$ *and* $(\bar{\chi})_n$ *are eigenstates of* $\lambda(\mathcal{J})$ *with eigenvalues* $-n$ *and* n *respectively.*

Proof. We prove the lemma by induction. For $n = 1$ we have

$$\lambda(\mathcal{J})(\chi) = -\chi\,, \quad \lambda(\mathcal{J})(\bar{\chi}) = \bar{\chi}\,,$$

as $\lambda(\mathcal{J})a_1 = i\,a_2$, and $\lambda(\mathcal{J})a_2 = -i\,a_1$. Assuming that

$$\lambda(\mathcal{J})(\chi)_{n-1} = -(n-1)(\chi)_{n-1}\,, \qquad \lambda(\mathcal{J})(\bar{\chi})_{n-1} = (n-1)(\bar{\chi})_{n-1}\,,$$

we find

$$\lambda(\mathcal{J})(\chi)_n = \lambda(\mathcal{J})(\chi)_{n-1}(\chi + n - 1) = -n\,(\chi)_n\,,$$

where we have used $\Delta\mathcal{J} = e^{-z\,P_2} \otimes \mathcal{J} + \mathcal{J} \otimes 1$ and $\lambda(e^{-z\,P_2})\,\chi = \chi + 1$. In similar way we get

$$\lambda(\mathcal{J})(\bar{\chi})_n = n\,(\bar{\chi})_n$$

observing that $\lambda(e^{-z\,P_2})\,\bar{\chi} = \bar{\chi} + 1$. ∎

(3.5) LEMMA. *The polynomial* $\rho_n = (\bar{\chi})_n\,(1-\chi)_n = (\chi)_n\,(1-\bar{\chi})_n$ *is invariant under the action of* $\lambda(\mathcal{J})$, *i.e.* $\lambda(\mathcal{J})\,\rho_n = 0$. *Moreover it can be written as* $\rho_n = \rho(\rho+2)(\rho+6)\cdots(\rho+n(n-1))$, *where* $\rho = \bar{\chi}(1-\chi)$.

Proof. We start by proving the second part of the lemma. Also in this case we adopt the induction technique. We have $\rho_1 = \rho$ and $\rho_n = (\bar{\chi})_{n-1}\,(\bar{\chi}+n-1)\,(1-\chi)_{n-1}\,(n-\chi)$. Since the relation $(\chi - \bar{\chi})P(\chi) = P(\chi+1)(\chi-\bar{\chi})$ holds for any polynomial $P(\chi)$, it is straightforward to commute $(\bar{\chi}+n-1)$ with $(1-\chi)_{n-1}$. As a result of the commutation we find

$$\rho_n = (\bar{\chi})_{n-1}\,(1-\chi)_{n-1}(\bar{\chi}\,(1-\chi) + n(n-1)) = \rho_{n-1}(\rho + n(n-1))$$

and then $\rho_n = \rho(\rho+2)(\rho+6)\cdots(\rho+n(n-1))$.

The proof of the first statement is a direct consequence of the previous result. Indeed from $(\bar{\chi}+\alpha)\,(1-\chi-\alpha) = (\chi+\alpha)\,(1-\bar{\chi}-\alpha)$, where α is any number, we find $(\bar{\chi})_n\,(1-\chi)_n = (\chi)_n\,(1-\bar{\chi})_n$. Moreover

$$\lambda(\mathcal{J})\,\rho_n = \lambda(\mathcal{J})\,\rho_{n-1}(\rho+n(n-1)) = \lambda(\mathcal{J})\rho_{n-1}\lambda(e^{-z\,P_2})(\rho+n(n-1))\,.$$

Due to the fact that $\lambda(\mathcal{J})\,\rho = 0$, we see that $\lambda(\mathcal{J})\,\rho_{n-1} = 0$ implies $\lambda(\mathcal{J})\,\rho_n = 0$. The lemma is therefore proved. ∎

We shall write the eigenstates of $\lambda(\mathcal{J})$ as

$$\psi_r = \sum_\ell c_r^\ell \, \rho_\ell \, (\bar\chi)_r \, , \qquad \psi_{-r} = \sum_\ell c_{-r}^\ell \, \rho_\ell \, (\chi)_r \, .$$

As a matter of fact, by a simple computation, we find

$$\lambda(\mathcal{J}) \, \psi_r = r \, \psi_r \, , \qquad \lambda(\mathcal{J}) \, \psi_{-r} = - \, r \, \psi_{-r} \, .$$

The coefficients c_r^ℓ of the expansion will be determined by using (3.1). It will appear that the choice of the polynomials ρ_n proves to be essential when we try to diagonalize the Casimir $\mathcal{C} = 4 \, H^+ H^-$. To this purpose we find it very useful to introduce an auxiliary element, which, according to the following proposition, yields a very simple form for the relations in $\mathcal{U}_\ell(E(2))$.

(3.6) PROPOSITION. *Let* $U^\pm = (1/2) \, (1 + e^{\mp z \, P_2} \mp iz \, e^{\mp z \, P_2/2} P_1)$ *and* $\mathcal{H}^\pm = U^\pm H^\pm$ *. Then*

$$[\mathcal{J}, \mathcal{H}^+] = \mathcal{H}^+ \, , \qquad [\mathcal{J}, \mathcal{H}^-] = -\mathcal{H}^- \, , \qquad [\mathcal{H}^+, \mathcal{H}^-] = 0 \, .$$

Moreover $\tilde{\mathcal{C}} = \mathcal{H}^+ \mathcal{H}^- = H^+ H^- (1 + z^2 H^+ H^-) \, .$

Proof. By direct computation. ∎

We will now discuss the action of $\lambda(\mathcal{H}^+)$ and $\lambda(\mathcal{H}^-)$ on ψ_r and ψ_{-r}. On the one hand, the proposition (3.6) and the involution property $(\mathcal{H}^+)^* = \mathcal{H}^-$ permit to write

$$\lambda(\mathcal{H}^+) \, \psi_{\pm r} = \epsilon \, \psi_{\pm r+1} \, , \qquad \lambda(\mathcal{H}^-) \, \psi_{\pm r} = \bar\epsilon \, \psi_{\pm r-1} \, . \tag{3.7}$$

On the other hand, the left hand side of (3.7) can direcly computed.

Indeed:

(3.8) LEMMA. *The following relations hold*

$$\lambda(\mathcal{H}^+)\psi_{-r} = -\frac{1}{z} \sum_\ell \left[c_{-r}^\ell \, (\ell + r) + c_{-r}^{\ell+1} (\ell + 1)(\ell + r)(\ell + r + 1) \right] \rho_\ell \, (\chi)_{r-1} \, ,$$

$$\lambda(\mathcal{H}^-)\psi_{-r} = -\frac{1}{z} \sum_\ell c_{-r}^{\ell+1} \, (\ell + 1) \, \rho_\ell \, (\chi)_{r+1} .$$

Proof. A straightforward calculation shows that

$$\lambda(U^+) \chi^n \bar\chi^m = \chi^n (\bar\chi + 1)^m \, , \quad \lambda(U^-) \bar\chi^m \chi^n = (\bar\chi - 1)^m \chi^n \, .$$

Using this result it is not difficult to complete the proof. $\blacksquare$

Equations (3.7) and (3.8) imply the recurrence relations

$$\begin{aligned}
\epsilon\, c^{\ell}_{-r+1} &= -\frac{1}{z}\left[c^{\ell}_{-r}\,(\ell+r) + c^{\ell+1}_{-r}\,(\ell+1)(\ell+r)(\ell+r+1)\right],\\
\bar{\epsilon}\, c^{\ell}_{-r-1} &= -\frac{1}{z} c^{\ell+1}_{-r}\,(\ell+1)\,.
\end{aligned}$$

It is easily verified that the coefficients

$$c^{\ell}_{-r} = (-kz/\bar{\epsilon})^r \frac{(kz^2)^\ell}{\ell!(\ell+r)!}\,,$$

solve the two recurrence relations, provided that k is related to $\epsilon\bar{\epsilon}$ by

$$|\epsilon|^2 = \epsilon\bar{\epsilon} = k(1+z^2\,k)\,.$$

Recalling that $\tilde{\mathcal{C}} = \mathcal{H}^+\mathcal{H}^- = H^+H^-(1+z^2H^+H^-)$, we get

$$\lambda(H^+H^-)\,\psi_{-r} = k\,\psi_{-r}\,.$$

We have therefore proved the following

(3.9) PROPOSITION. *The states that diagonalize $\lambda(\mathcal{J})$ and $\lambda(H^+H^-)$ are*

$$\psi_{-r} = \sum_{\ell=0}^{\infty} c^{\ell}_{-r}\,\rho_\ell\,(\chi)_r\,, \qquad \psi_r = \sum_{\ell=0}^{\infty} c^{\ell}_{r}\,\rho_\ell\,(\bar{\chi})_r\,,$$

where

$$c^{\ell}_{-r} = (-kz/\bar{\epsilon})^r \frac{(kz^2)^\ell}{\ell!(\ell+r)!}\,, \qquad c^{\ell}_{r} = (-kz/\epsilon)^r \frac{(kz^2)^\ell}{\ell!(\ell+r)!}\,.$$

Some final remarks are in order. We observe that also in the angular momentum basis the states $\psi_{\pm r}$ can be written, as *classical* hypergeometric series with noncommutative coefficients. Indeed, by making explicit the form of ρ_ℓ we get:

$$\begin{aligned}
\psi_{-r} &= \sum_{\ell=0}^{\infty}(-kz/\bar{\epsilon})^r \frac{(kz^2)^\ell}{\ell!(\ell+r)!}(\bar{\chi})_\ell\,(1-\chi)_\ell\,(\chi)_r\\
&= {}_2F_1\left[\begin{matrix}\bar{\chi} & 1-\chi\\ r+1 & \end{matrix}\;;\; kz^2\right]\,(-kz/\bar{\epsilon})^r\,\frac{(\chi)_r}{r!}\,,
\end{aligned}$$

and

$$\psi_r = \; {}_2F_1 \left[\begin{matrix} \bar\chi \qquad 1-\chi \\ r+1 \end{matrix} \; ; \; kz^2 \right] \, (-kz/\epsilon)^r \, \frac{(\bar\chi)_r}{r!} \; .$$

We want to stress that the situation is very different from the deformation treated in [31]. The *quantum* deformation is signified through the noncommutative variables χ and $\bar\chi$ that appear in the coefficients a_1 and a_2 of ${}_2F_1$. The coefficient b_1 and the hypergeometric variable ζ are, instead, numbers. The classical limit $z \to 0$ of the hypergeometric ${}_2F_1$ yield again the usual Bessel functions in the commutative variable $\bar x x$. This is not surprising. It is nevertheless rather peculiar that this result is due to a kind of confluence phenomenon caused by the non commutativity of the arguments.

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References.

1 . Kirillov A.A., "*Elements of the Theory of Representations*", (Springer Verlag, Berlin, 1990).

2 . Podleś P., *Lett. Math. Phys.*, **14** (1987) 193.

3 . Vaksman L.L. and Soibelman Y.S., *Funct. Anal. Appl.*, **22** (1988) 170.

4 . Masuda T., Mimachi K., Nakagami Y., Noumi M., Ueno K., *J. Funct. Anal.*, **99** (1991) 127.

5 . Noumi M., Mimachi K., "*Askey-Wilson polynomials as spherical functions on $SU_q(2)$*", in Lecture Notes in Mathematics n. 1510, 221, (Kulish P.P. ed., Springer–Verlag, 1992).

6 . Koornwinder T.H., *Proc. Kon. Ned. Akad. Wet.*, Series A, **92** (1989) 97.

7 . Koelink H.T., Koornwinder T.H., *Proc. Kon. Ned. Akad. Wet.*, Series A, **92** (1989) 443.

8 . Vilenkin N.Ja. and Klimyk A.U., "*Representation of Lie groups and Special Functions*", Vol. 3, (Kluwer Acad. Publ., Dordrecht, 1992).

9 . Noumi M., Yamada H., Mimachi K., "Finite-dimensional representations of the quantum group $GL_q(n,\mathbf{C})$ and the zonal spherical functions on $U_q(n–1)/U_q(n)$", *Japanese J. Math.* (to appear).

10 . Celeghini E., Giachetti R., Sorace E. and Tarlini M., *J. Math. Phys.*, **31** (1990) 2548 .

11 . Celeghini E., Giachetti R., Sorace E. and Tarlini M., *J. Math. Phys.*, **32** (1991) 1155.

12 . Bonechi F., Giachetti R., Sorace E. and Tarlini M.,*Commun. Math. Phys*, **169** (1995) 463.

13 . Ballesteros A., Celeghini E., Giachetti R., Sorace E. and Tarlini M. *J. Phys. A: Math. Gen.*, **26** (1993) 7495.

14 . Vaksman L.L. and Korogodski L.I., *Sov. Math. Dokl.*, **39** (1989) 173.

15 . Koelink H.T., "*On quantum groups and q-special functions*", Thesis (Leiden University, 1991) and *Duke Math. J.*, **76** (1994) 483.

16 . Baaj S., *C.R. Acad. Sci. Paris*, **314** (1992) 1021.

17 . Woronowicz S.L., *Lett. Math. Phys.*, **23** (1991) 251.

18 . Woronowicz S.L., *Commun. Math. Phys.*, **144** (1992) 417.

19 . Woronowicz S.L., *Commun. Math. Phys.*, **149** (1992) 637.

20 . Schupp P., Watts P. and Zumino B., *Lett. Math. Phys.*, **24** (1992) 141.

21 . Bonechi F., Celeghini E., Giachetti R., Pereña C.M., Sorace E. and Tarlini M., *J. Phys. A: Math. Gen.*, **27** (1994) 1307.

22 . Ciccoli N. and Giachetti R., *Lett. Math. Phys.* **34** (1995) 37.

23 . Dabrowski L. and Sobczyk J., "Left regular representation and contraction of $sl_q(2)$ to $e_q(2)$", Preprint IFT-864/94.

24 . Celeghini E., Giachetti R., Sorace E. and Tarlini M., "*Contractions of quantum groups*", in Lecture Notes in Mathematics n. 1510, 221, (Kulish P.P. ed., Springer–Verlag, 1992).

25 . Ballesteros A., Herranz F.J., del Olmo M.A. and Santander M., *J. Phys. A: Math. Gen.*, **26** (1993) 5801.

26 . Ballesteros A., Herranz F.J., del Olmo M.A. and Santander M., *J. Math. Phys.*, **36** (1995) 631.

27 . Celeghini E., Giachetti R., Sorace E. and Tarlini M., *J. Math. Phys.*, **32** (1991) 1159.

28 . Ballesteros A., Herranz F.J., del Olmo M.A. and Santander M., *J. Phys. A: Math. Gen.*, **27** (1994) 1283.

29 . Lukierski J., Ruegg H., Nowicki A. and Tolstoy V.N. *Phys. Lett. B*, **264** (1991) 331.

30 . Ballesteros A., Herranz F.J., del Olmo M.A. and Santander M., *J. Math. Phys.*, **35** (1994) 4928.

31 . Bonechi F., Ciccoli N., Giachetti R., Sorace E. and Tarlini M., *Commun. Math. Phys.*, **175** (1996) 161.

32 . Manin Yu.I., "*Quantum groups and noncommutative geometry*" in Publications of C.R.M. 1561 (University of Montreal, 1988).

33 . Rieffel M.A., "*Deformation Quantization for actions of R^d*, Memoirs A.M.S., 506 (1993).

34 . Schmüdgen K., *Commun. Math. Phys.*, **159** (1994) 159.

35 . Ciccoli N. "*Quantum Planes and Quantum Cylinders from Poisson Homogeneous Spaces*", *J. Phys. A: Math. and Gen.*, in press.

36 . Koelink H.T., "*On quantum groups and q-special functions*", Thesis (Leiden University, 1991) and *Duke Math. J.*, **76** (1994) 483.

37 . Baaj S., *C.R. Acad. Sci. Paris*, **314** (1992) 1021.

38 . Bonechi F., Celeghini E., Giachetti R., Sorace E. and Tarlini M., *Phys. Rev. Lett.*, **68** (1992) 3718.

39 . Bonechi F., Celeghini E., Giachetti R., Sorace E. and Tarlini M., *Phys. Rev. B*, **32** (1992) 5727.

40 . Bonechi F., Celeghini E., Giachetti R., Sorace E. and Tarlini M., *J. Phys. A*, **25** (1992) L939.

41 . Dijkhuizen M.S., Koornwinder T.H., *Geom. Dedicata.*, **52** (1994) 291.